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Scaling behavior of the absorbing phase transition in a conserved lattice gas around the upper critical dimension

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We analyze numerically the critical behavior of a conserved lattice gas that was recently introduced as an example of the new universality class of absorbing phase transitions with a conserved field [Phys. Rev. Lett. **85**, 1803 (2000)]. We determine the critical exponent of the order parameter as well as the critical exponent of the order parameter fluctuations in $D=2,3,4,5$ dimensions. A comparison of our results and those obtained from a mean-field approach and a field theory suggests that the upper critical dimension of the absorbing phase transition is four.

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I. INTRODUCTION

The scaling behavior of directed percolation is recognized as the paradigmatic example of the critical behavior of several nonequilibrium systems that exhibits a continuous phase transition from an active state to an absorbing nonactive state (see for instance [1,2]). Such systems are common in physics, biology, as well as catalytic chemical reactions. This widespread occurrence corresponds to the well known universality hypothesis of Janssen and Grassberger that models, which exhibit a continuous phase transition to a single absorbing state generally belong to the universality class of directed percolation [3,4].

Recently Rossi *et al.* introduced a conserved lattice gas (CLG) with a stochastic short range interaction that exhibits a continuous phase transition to an absorbing state at a critical value of the particle density [5]. The CLG model is expected to belong to a new universality class of absorbing phase transitions characterized by a conserved field. Similar to the above hypothesis the authors conjectured that ''all stochastic models with an infinite number of absorbing states in which the order parameter evolution is coupled to a nondiffusive conserved field define a unique universality class'' [5]. Besides the CLG model the authors considered the conserved threshold transfer process model as well as a modification of the stochastic sandpile model of Manna [6] and observed numerically compatible values of the critical exponents. Furthermore a reaction-diffusion model was introduced in [7] that is expected to belong to the same universality class. This reaction-diffusion model allows to derive a field theoretical description that is expected to represent the critical behavior of the whole universality class [7].

In this work we consider for the first time the scaling behavior of the CLG model and therefore the critical behavior of the new universality class in higher dimensions. We determine numerically the critical exponent of the order parameter as well as the exponent of the order parameter fluctuations. Our results show that the values of the exponents depend on the dimension for $D\leqslant 4$. Above this dimension we observe a mean-field scaling behavior. Thus our results suggest that the upper critical dimension of the CLG model is four as already predicted from the field theoretical approach [7].

II. $D=2$

We consider the CLG model on D-dimensional cubic lattices of linear size L. Initially one distributes randomly $N=\rho L$ particles on the system where ρ denotes the particle density. In order to mimic a repulsive interaction a given particle is considered as *active* if at least one of its $2D$ neighboring sites on the cubic lattice is occupied by another particle. If all neighboring sites are empty the particle remains *inactive*. Active particles are moved in the next update step to one of their empty nearest neighbor sites, selected at random. Starting from a random distribution of particles the system reach after a transient regime a steady state that is characterized by the density of active sites ρ_{a}. The density ρ_{a} is the order parameter of the absorbing phase transition, i.e., it vanishes if the control parameter ρ is lower than the critical value ρ_{c}. In contrast to the work of Rossi *et al.* [5], who used a parallel update scheme, we applied in our simulations a random sequential update, i.e., all active sites are listed, and then updated in a randomly chosen sequence.

We consider in two dimensions simple cubic systems of linear size $L=32,64,128,\ldots,2048$ with periodic boundary conditions. Starting with a random configuration of particles, a sufficient number of update steps has to be performed to reach the steady state where the number of active sites fluctuates around an average value (see Fig. 1). Approaching the transition point, more and more update steps are needed to reach this steady state. For instance in the case $L=2048$ we use 2×10^6 update steps to ''equilibrate'' the system. In the steady state the number of active sites is monitored for 5×10^5 update steps. This procedure is repeated in all dimensions for at least ten different initial configurations. From this data we determine the average density of active sites $\langle\rho_{\mathrm{a}}\rangle$ as well as its fluctuations

$$\Delta\rho_{\mathrm{a}}=L^D(\langle\rho_{\mathrm{a}}^2\rangle-\langle\rho_{\mathrm{a}}\rangle^2). \tag{1}$$

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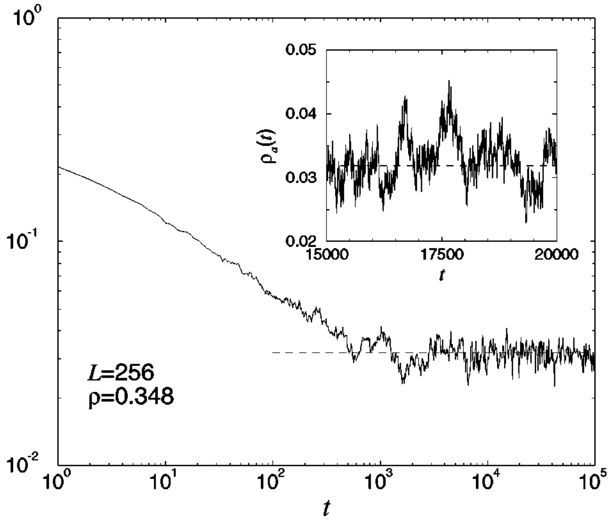

FIG. 1. The density of active sites ρ_a as a function of time (number of update steps) for a certain value of ρ. After a transient regime, which depends on the initial configuration, the density of active sites fluctuates around the steady state value $\langle\rho_a\rangle$ (dashed line).

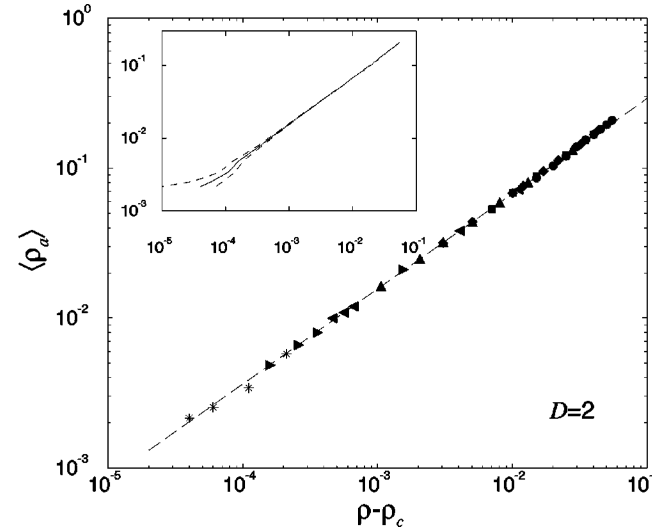

FIG. 3. The average density of active sites $\langle\rho_a\rangle$ as a function of $\rho-\rho_c$ in $D=2$. The symbols mark different system sizes L (see Fig. 2). The dashed line corresponds to a power-law fit with $\rho_c=0.344\,94\pm0.000\,03$ and $\beta=0.637\pm0.009$. In the inset we display the same data for $\rho_c=0.344\,91$ and $\rho_c=0.344\,97$. Compared to the above value (solid line) both curves are characterized by significant curvatures in the plotted log-log diagram. For the sake of simplicity we plot in the inset lines instead of symbols.

In Fig. 2 we plot the average density of active sites as a function of the particle density ρ for various system sizes L. As one can see $\langle\rho_a\rangle$ tends to zero in the vicinity of $\rho\approx0.345$. Assuming that the scaling behavior of the density of active sites is given by

$$\langle\rho_a\rangle\sim(\rho-\rho_c)^{\beta}, \tag{2}$$

one varies ρ_c until one gets a straight line in a log-log plot. Convincing results are obtained for $\rho_c=0.344\,94\pm0.000\,03$ and the corresponding curve is shown in Fig. 3. For $\rho_c=0.344\,91$ and $\rho_c=0.344\,97$ we observe significant curvatures in the log-log plot (see inset of Fig. 3). In this way we estimate the error bars in the determination of the critical density. A regression analysis yields the value of the order parameter exponent $\beta=0.637\pm0.009$. Rossi *et al.* reported the values $\beta=0.63\pm0.01$ and $\rho_c=0.288\,75$, obtained from simulations with a parallel update scheme and of smaller system sizes ($L\leqslant512$) [5]. Thus we see that the different update scheme affects only the value of the critical density but not the critical exponent β. Furthermore the value of β differs from the corresponding value of the directed percolation universality class $\beta_{DP}=0.584\pm0.004$ (see [2]).

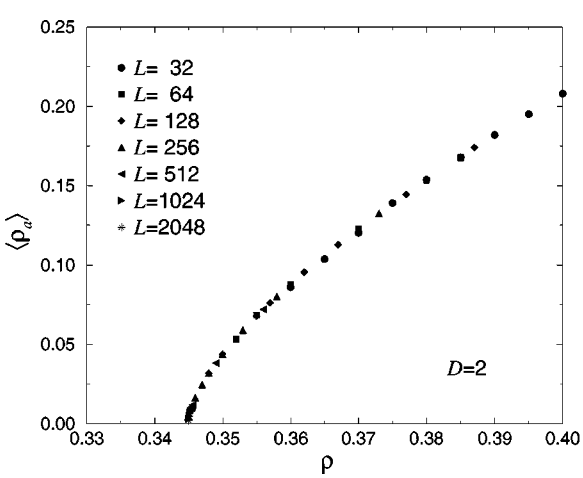

FIG. 2. The average density of active sites $\langle\rho_a\rangle$ as a function of the global particle density ρ for various system sizes L in $D=2$ dimensions.

The fluctuations of the order parameter [Eq. (1)] are plotted in Fig. 4. Approaching the transition point $\Delta\rho_a$ increases and diverges at ρ_c. Close to the critical point the fluctuations scale as

$$\Delta\rho_a\sim(\rho-\rho_c)^{-\gamma} \tag{3}$$

in the active phase (see inset of Fig. 4). Using a regression analysis one gets $\gamma=0.384\pm0.023$.

III. D=3

For the three dimensional model system sizes from $L=16$ to $L=160$ are considered. Close to the transition point 5×10^6 update steps are used to reach the steady state in the case of the largest system size. The obtained results for the density of active sites $\langle\rho_a\rangle$ are shown in Fig. 5. A straight line in a log-log plot is obtained for $\rho_c=0.2179\pm0.0001$ and the corresponding value of the order parameter exponent is $\beta=0.837\pm0.015$. The error bars of the critical density are determined in the same way as for $D=2$. The value of the order parameter exponent agrees within the error bars with the corresponding values of the Manna sandpile model ($\beta=0.84\pm0.02$) and of a reaction diffusion model ($\beta=0.86\pm0.02$) that are expected to belong to the same universality

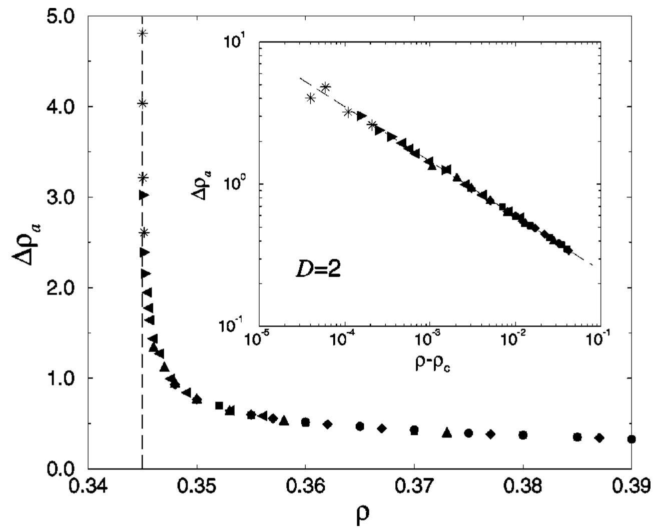

FIG. 4. The fluctuations of the order parameter $\Delta\rho_a$ as a function of the global density ρ in $D=2$. The symbols mark different system sizes (see Fig. 2) and the dashed line corresponds to the critical density $\rho_c=0.344\,94$. The inset displays the fluctuations as a function of $\rho-\rho_c$. The dashed line corresponds to a power-law behavior [Eq. (3)] with an exponent $\gamma=0.384\pm0.023$.

class [7]. Our result differs slightly from the corresponding value of the directed percolation universality class $\beta_{DP}=0.81\pm0.01$ (see [2]).

Similar to the two dimensional case the order parameter fluctuations $\Delta\rho_a$ display a maximum at the transition point but the dependence of $\Delta\rho_a$ on the density of particles is not clear. In Fig. 6 we plot the fluctuations in a log-log plot. It seems that the asymptotic behavior corresponds to a power law [Eq. (3)] with an exponent $\gamma=0.18\pm0.06$. But as the inset of Fig. 6 shows the data are also consistent with the assumption that the fluctuations are characterized by a logarithmic divergence (i.e., $\gamma=0$) at ρ_c according to

$$\Delta\rho_a \sim |\ln(\rho-\rho_c)|. \tag{4}$$

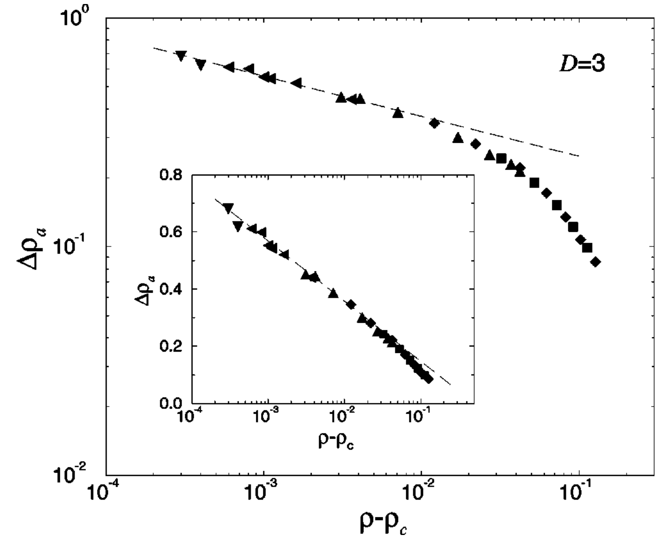

FIG. 6. The fluctuations of the order parameter $\Delta\rho_a$ as a function of $\rho-\rho_c$ in $D=3$. The symbols mark different system sizes (see Fig. 5). The data can be interpreted either as a power law with a small exponent γ (the dashed line corresponds to $\gamma=0.18\pm0.06$) or as a logarithmic growth [see Eq. (4) and inset, where the dashed line is to guide the eye].

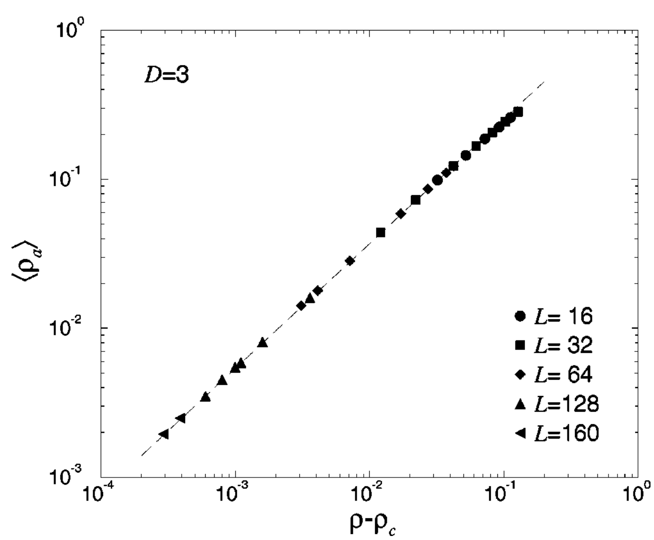

FIG. 5. The average density of active sites $\langle\rho_a\rangle$ as a function of $\rho-\rho_c$ for various system sizes L in $D=3$. The dashed line corresponds to a power-law fit with $\rho_c=0.2179\pm0.0001$ and $\beta=0.837\pm0.015$.

Thus the numerical data indicate that the fluctuations $\Delta\rho_a$ diverge at the critical point but the data can be interpreted either as a power law with a small exponent or as a logarithmic growth. Further investigations are needed to clarify this point.

IV. $D=4$

In order to analyze the scaling behavior of the four dimensional CLG model we performed numerical simulations with system sizes $L\in\{8,16,32,48\}$. In the case of the largest system size 6×10^6 updates steps were used to reach the steady state. Plotting the values of the average particle density $\langle\rho_a\rangle$ in a log-log plot, no straight line, i.e., no pure power law behavior could be observed. To illustrate this behavior we plot in Fig. 7 the logarithmic derivative

$$\frac{\partial \ln\langle\rho_a\rangle}{\partial \ln(\rho-\rho_c)}, \tag{5}$$

which can be interpreted as an effective exponent β_{eff}. If the scaling behavior of the active site density is given by Eq. (2) the logarithmic derivative tends to the value of β for $\rho-\rho_c\to0$. This behavior is observed in the three-dimensional case (see Fig. 7). For $D=4$ the logarithmic derivative displays no saturation for $\rho\to\rho_c$, i.e., the scaling behavior of the four-dimensional model cannot be described by a simple power-law behavior. Significant corrections to the usual scaling behavior [Eq. (2)] occur for instance at the upper critical

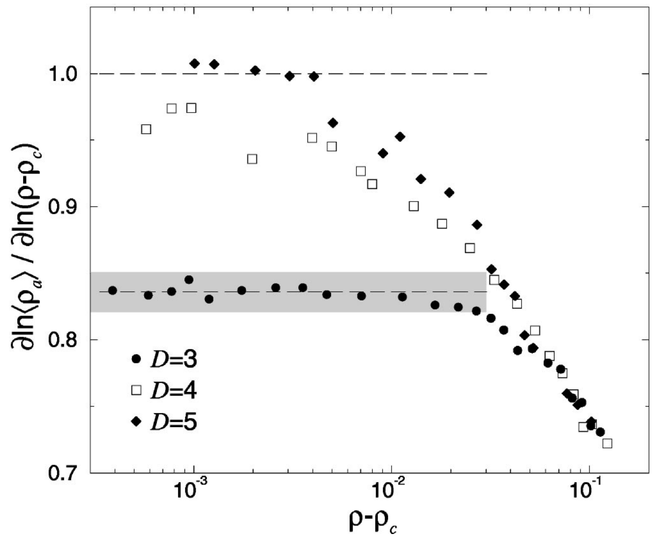

FIG. 7. The logarithmic derivative [Eq. (5)] as a function of $\rho-\rho_c$. The logarithmic derivative can be interpreted as an effective exponent β_{eff}. The figure shows that the four-dimensional exponent does not display a saturation as the exponents of the three- and five-dimensional models do. The dashed lines correspond to the three-dimensional value $\beta=0.837\pm0.015$ and the mean-field value $\beta=1$, respectively. The shadowed region marks the uncertainty of the determination of β.

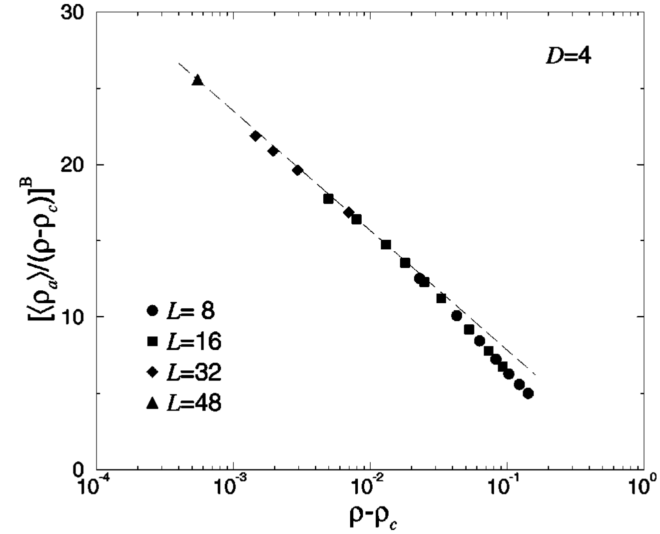

FIG. 8. The density of active sites in $D=4$ rescaled according to Eq. (6). The assumed asymptotic behavior (straight line in a log-log plot) is obtained for $\rho_c=0.1571\pm0.0002$ and $B=0.39$. Thus the scaling behavior of the order parameter of the four-dimensional model is governed by the mean-field exponent $\beta=1$ modified by logarithmic corrections. The dashed line is just to guide the eye.

dimensional where the scaling behavior is governed by the mean-field exponents modified by logarithmic corrections.

Recently a modified version of the CLG model was introduced where the active particles are distributed to randomly chosen empty lattice sites [8]. Since the randomness of the particle hopping breaks long range spatial correlations this model is expected to be characterized by the mean-field scaling behavior of the CLG model. Mapping the dynamics of this random hopping CLG model to a simple branching process one can derive the critical exponent $\beta=1$. This value of the order parameter exponent is also obtained from a field theoretical description of the CLG model [9] and was already predicted from a phenomenological field theory [10] of the so-called fix-energy stochastic sandpile model that is expected to be in the same universality class.

TABLE I. The critical density ρ_c and the critical exponents β and γ of the CLG model for various dimensions D. The symbol * denotes logarithmic corrections to the power-law behavior. In the case of the three-dimensional model the data of the fluctuation could be interpreted as a small exponent or as a logarithmic growth (see text).

D	ρ_c	β	γ
2	0.34494 ± 0.00003	0.637 ± 0.009	0.384 ± 0.023
3	0.2179 ± 0.0001	0.837 ± 0.015	0 or 0.18 ± 0.06
4	0.1571 ± 0.0002	1*	0*
5	0.1230 ± 0.0004	1	0

Thus we assume that the scaling behavior of the order parameter is given in leading order by the ansatz

$$\langle\rho_a\rangle\sim(\rho-\rho_c)^{\beta}|\ln(\rho-\rho_c)|^{B} \qquad (6)$$

with $\beta=1$. Therefore we varied in our analysis B and ρ_c until we get the expected asymptotic behavior. The best result is obtained for $B=0.39$ and $\rho_c=0.1571\pm0.0002$ and the corresponding scaling plot is shown in Fig. 8. As one can see our data are consistent with the assumption that the asymptotic scaling behavior of the four-dimensional model obeys Eq. (6).

The mean-field behavior of the fluctuations is characterized by $\gamma=0$ that corresponds to a jump [8]. Taking the logarithmic corrections into account we assume that the asymptotic scaling behavior of the fluctuations obeys the ansatz

$$\Delta\rho_a=\Delta\rho_a^{(0)}-\mathrm{const}(\rho-\rho_c)|\ln(\rho-\rho_c)|^{\Gamma}. \qquad (7)$$

Plotting the fluctuations as a function of $(\rho-\rho_c)|\ln(\rho-\rho_c)|^{\Gamma}$ one varies the correction exponent Γ until one gets a straight line. Convincing results are observed for $\Gamma=1.66\pm0.1$ and the corresponding plot is shown in Fig. 9 (for $\Gamma>1.76$ and $\Gamma<1.56$ we observe significant curvatures). The extrapolation to the vertical axis yield the fluctuations at the transition point $\Delta\rho_a^{(0)}=0.325\pm0.005$.

Thus we get that the scaling behavior of the four-dimensional CLG model is characterized by the mean-field exponents modified by logarithmic corrections and we conclude therefore that the upper critical dimension of the CLG model is $D_c=4$. This value agrees with the conjecture of a field theory [7].

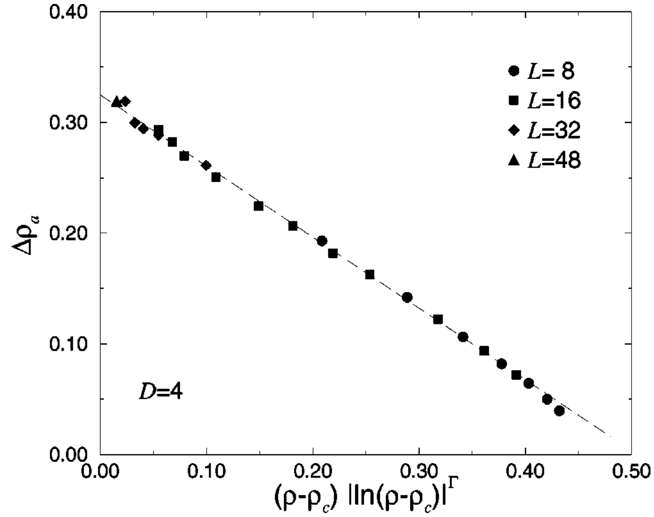

FIG. 9. The fluctuations of the order parameter $\Delta\rho_{\mathrm{a}}$ as a function of $(\rho-\rho_{\mathrm{c}})|\ln(\rho-\rho_{\mathrm{c}})|^{\Gamma}$ in $D=4$ [see Eq. (7)]. Nearly straight lines are obtained for $\Gamma=1.66\pm0.1$. Thus the scaling behavior of the order parameter of the four-dimensional model is governed by the mean-field exponent $\gamma=0$ modified by logarithmic corrections. The dashed line corresponds to a linear fit with the slope -0.643 and the intercept $\Delta\rho_{\mathrm{a}}^{(0)}=0.325$.

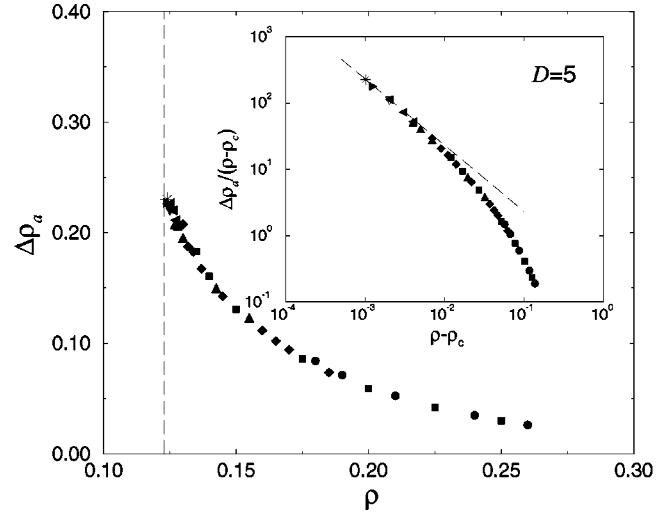

FIG. 11. The fluctuations of the order parameter $\Delta\rho_{\mathrm{a}}$ as a function of the density ρ in $D=5$. The symbols mark different system sizes L (see Fig. 10). The dashed line marks the value of the critical density. The inset displays $\Delta\rho_{\mathrm{a}}/(\rho-\rho_{\mathrm{c}})$ as a function of $\rho-\rho_{\mathrm{c}}$. The asymptotic behavior is characterized by a simple $(\rho-\rho_{\mathrm{c}})^{-1}$ dependence (dashed line) that corresponds to the mean-field behavior (see text).

V. $D=5$

In the case of the five-dimensional model we considered system sizes from $L=4$ up to $L=18$. In the latter case 10^7 update steps were used to reach the steady state close to the transition point. The obtained values of the order parameter are plotted in Fig. 10. The average density of active sites seems to vanish linearly at the transition point. This is supported by the logarithmic derivative [Eq. (5)] that is plotted in Fig. 7. The effective exponent saturates in the vicinity of the mean-field solution $\beta=1$. One has to admit that the small available system sizes in $d=5$ ($L\leqslant18$) hinder an analysis of the pure critical scaling behavior. But as we will see the data are sufficient to indicate that the asymptotic scaling behavior is given by $\beta=1$. Therefore, we plot in the inset of Fig. 10 $\langle\rho_{\mathrm{a}}\rangle/(\rho-\rho_{\mathrm{c}})$ as a function of $\rho-\rho_{\mathrm{c}}$. Approaching the transition point the rescaled density of active sites indeed saturates. Thus the five-dimensional CLG model is characterized by the mean-field scaling behavior

$$\langle\rho_{\mathrm{a}}\rangle\sim\rho-\rho_{\mathrm{c}}. \tag{8}$$

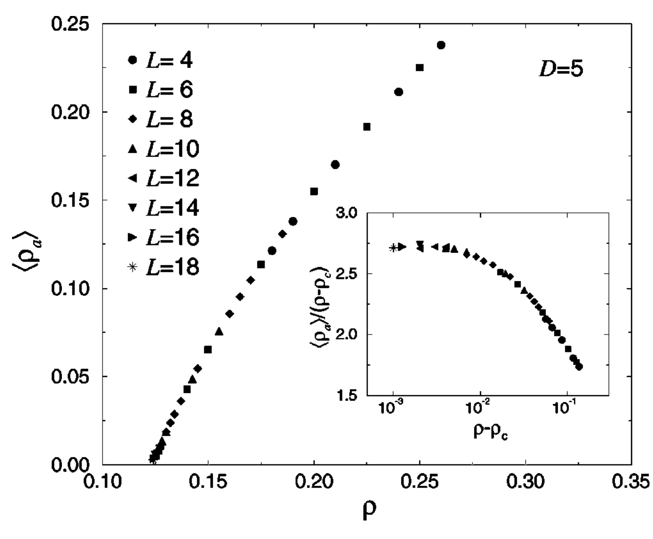

FIG. 10. The average density of active sites $\langle\rho_{\mathrm{a}}\rangle$ as a function of the density ρ for various system sizes L in $D=5$. The inset displays $\langle\rho_{\mathrm{a}}\rangle/(\rho-\rho_{\mathrm{c}})$ as a function of $\rho-\rho_{\mathrm{c}}$. The saturation for $\rho-\rho_{\mathrm{c}}\to0$ indicates that the asymptotic behavior of the order parameter agrees with the mean field behavior $\langle\rho_{\mathrm{a}}\rangle\sim\rho-\rho_{\mathrm{c}}$. The critical density is $\rho_{\mathrm{c}}=0.1230\pm0.0004$.

The fluctuations of the order parameter $\Delta\rho_{\mathrm{a}}$ are plotted in Fig. 11. As one can see the fluctuations are characterized by a jump at the transition point. Following the mean-field behavior one expects that the asymptotic behavior of the fluctuations is given by

$$\Delta\rho_{\mathrm{a}}=\Delta\rho_{\mathrm{a}}^{(0)}-\mathrm{const}(\rho-\rho_{\mathrm{c}}). \tag{9}$$

In order to confirm this ansatz we plot $\Delta\rho_{\mathrm{a}}/(\rho-\rho_{\mathrm{c}})$ as a function of $\rho-\rho_{\mathrm{c}}$. If the above ansatz is valid the corresponding curves have to display an asymptotic power-law behavior with the exponent -1 and the prefactor $\Delta\rho_{\mathrm{a}}^{(0)}$. This behavior is indeed observed (see inset of Fig. 11) and we get $\Delta\rho_{\mathrm{a}}^{(0)}=0.231\pm0.006$.

Thus the asymptotic behavior of the five-dimensional CLG model is characterized by the mean-field exponents. This behavior strongly supports the above conclusion that the upper critical dimension of the CLG model is four.

VI. CONCLUSIONS

We analyze numerically the critical behavior of a CLG model in various dimensions. The values of the critical density and of the critical exponents are determined and the results are listed in Table I. Our analysis suggests that four is the upper critical dimension of the CLG model, i.e., the critical exponents depend on the value of the dimension for $D \leqslant 4$. Above this value we observe a mean-field like scaling behavior.

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